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Krull Dimension of the Enveloping Algebra of $sl(2, \mathbb{C})$

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1. INTRODUCTION

Let U denote the enveloping algebra of the simple Lie algebra $sl(2, \mathbb{C})$. In this paper it is shown that the Krull dimension of U (denoted $|U|$) is two.

If $U(g)$ is the enveloping algebra of a finite-dimensional solvable Lie algebra g then it is straightforward to show that $|U(g)| = \dim g$ [5, 3.8.11]. The problem as to the Krull dimension of U was first mentioned by Gabriel and Nouazé [9] — they show that U has a chain of prime ideals of length two, and none of length greater than two. From this they conclude that the Krull dimension of U is two, although the correct conclusion is only that $|U| \geqslant 2$. Subsequent to [9], both Arnal and Pinczon [1] and Roos [10] established that if R were a non-artinian simple primitive factor ring of U then $|R| = 1$. More recently the author [11] proved that if R were a non-artinian primitive factor ring of U which was not simple then again $|R| = 1$. The result in the present paper implies those in [1, 10, 11].

The fundamental tool in the proof that $|U| = 2$ is Gelfand–Kirillov dimension (GK-dimension). The proof is in two parts. In Section 2 a number of preliminary results (already known) concerning GK-dimension are recalled. In particular, Lemma 2.3 provides the basic connection between GK-dimension and Krull dimension. The more detailed analysis of U is carried out in Section 3. The crucial result is that any finitely generated U module of Krull dimension 1 has GK-dimension 2 — the result then quickly follows from Lemma 2.3.

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2. Gelfand–Kirillov Dimension

For the basic definitions and properties concerning GK-dimension the reader is referred to [3, 8]. We present here only those properties which are essential for our purposes. For the rest of Section 2 let R denote a factor ring of the enveloping algebra of a finite dimensional Lie algebra.

As in [8], given a finitely generated R-module M (all modules will be left modules), we can associate a polynomial $q(M)$ (the Hilbert–Samuel polynomial) with M such that the degree of q is precisely the GK-dimension of M (denoted by $GK(M)$). Set $e(M)$ to be $GK(M)! \times$ (leading coefficient of $q(M)$). Recall that $e(M)$ is a positive integer.

Lemma 2.1 [8, Lemma 2.2]. *Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of finitely generated R-modules. Then one of the following holds*:

(i) $GK(M_1) < GK(M)$ *and* $GK(M_2) = GK(M)$ *and* $e(M_2) = e(M)$;

(ii) $GK(M_1) = GK(M) = GK(M_2)$ *and* $e(M) = e(M_1) + e(M_2)$;

(iii) $GK(M_2) < GK(M)$ *and* $GK(M_1) = GK(M)$ *and* $e(M_1) = e(M)$.

Corollary 2.2. (i) *For any submodule N of a finitely generated R-module, M, $GK(M) = \max\{GK(M/N), GK(N)\}$.* (ii) *If M is an R-module with $GK(M) = d$, and $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is a chain of submodules satisfying $GK(M_i/M_{i+1}) = d$, then the chain has at most $e(M)$ terms.*

Proof. (i) follows immediately from the lemma. By (i) and induction, $GK(M_i) = d$ for each i. Then (ii) follows by repeatedly applying (ii) of the lemma.

The following lemma is implicit in [7, 2.2] but does not appear to have been stated explicitly anywhere.

Lemma 2.3. *Suppose that for all finitely generated R-modules M' with $|M'| = \alpha$ that $GK(M') \geqslant \alpha + r$ $(\alpha, r \in \mathbb{N})$. Let M be a finitely generated R-module with $|M| \geqslant \alpha$. Then $GK(M) \geqslant |M| + r$.*

Proof. By induction on $|M|$. It is true by the hypothesis when $|M| = \alpha$. Suppose the result is true for modules with Krull dimension strictly less than β, and let $|M| = \beta$. Then there exists a chain $M = M_0 \supseteq M_1 \supseteq M_2 \cdots$ of submodules such that $|M_i/M_{i+1}| = \beta - 1$ for $i = 0, 1, 2, \ldots$. By the induction hypothesis, $GK(M_i/M_{i+1}) \geqslant \beta - 1 + r$. It follows from Corollary 2.2 that $GK(M) > \beta - 1 + r$; that is $GK(M) \geqslant \beta + r$.

Two interesting consequences of this lemma are worth mentioning:

(1) [7, 2.2]. Bernstein [2] has shown for the Weyl algebra A_n that any simple A_n-module has GK-dimension at least n. So applying the lemma with $\alpha = 0$, $r = n$ it follows in particular that $|A_n| \leqslant n$ (because $GK(A_n) = 2n$).

(2) If R is a simple non-artinian factor ring of an enveloping algebra with $GK(R) = 2$, then $|R| = 1$. To see this let M be a simple R-module. Then M is not finite dimensional (otherwise R itself is finite dimensional and then $GK(R) = 0$), so $GK(M) \geqslant 1$. Hence the lemma implies, with $\alpha = 0$, $r = 1$, $|R| \leqslant 1$. In particular this argument gives a brief proof of the fact that the simple primitive factor rings of U which are not artinian have Krull dimension 1 — see [1, 10].

An unpublished result of the author actually shows that such a ring R (i.e., a factor of an enveloping algebra), if it is primitive, cannot have GK-dimension 1.

Finally we give a particularly easy lemma which we will need.

LEMMA 2.4. *Let I be a left ideal of R, and let S be a subring of R such that S is a finitely generated algebra over $\mathbb{C}$, and such that $I \cap S = 0$. Then $GK(R/I) \geqslant GK(S)$.*

Proof. Let $V \supseteq \mathbb{C}$ be a finite-dimensional generating subspace of S, and let $W \supseteq V$ be a finite-dimensional generating subspace of R. Then

$$GK(R/I) = \limsup_{n\to\infty} \frac{\log \dim((I + W^n)/I)}{\log n}$$

$$\geqslant \limsup_{n\to\infty} \frac{\log \dim((I + V^n)/I)}{\log n}$$

but as $V^n \subseteq S$ and $I \cap S = 0$, $\dim((I + V^n)/I) = \dim V^n$. So

$$GK(R/I) \geqslant \limsup_{n\to\infty} \frac{\log \dim V^n}{\log n} = GK(S).$$

3. MAIN RESULT

We begin with some notation and elementary facts about $sl(2, \mathbb{C})$. More detail may be found in Dixmier [4]. We take as a basis for $sl(2, \mathbb{C})$ the elements e, f, h subject to the relations

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

The element $Q = 4ef + h^2 - 2h = 4fe + h^2 + 2h$ is central in U. Given $n \in \mathbb{N}$, U has a unique finite-dimensional simple module of dimension $(n + 1)$. This module is annihilated by the central element $Q - n(n + 2)$.

Let $n \in \mathbb{N}$. The simple module of dimension $(n + 1)$ may be thought of as a $\mathbb{C}$-vector space with basis $1, f, f^2, \ldots, f^n$ where the action of $sl(2, \mathbb{C})$ on these basis elements is given as follows:

$$e \cdot f^j = j(n-j+1) f^{j-1} \qquad \text{with} \quad e \cdot 1 = 0;$$
$$f \cdot f^j = f^{j+1} \qquad \text{with} \quad f \cdot f^n = 0;$$
$$h \cdot f^j = (n-2j) f^j.$$

This may be deduced from (for example) [5, 7.2.7].

LEMMA 3.1. *Let U/J be an artinian U-module such that each composition factor is isomorphic to the same finite-dimensional simple module S, say. Then U/J is of length at most* $\dim_{\mathbb{C}} S$.

Proof. Let $P = \operatorname{ann}(S)$. Then U/P is simple artinian with simple module S, so U/P is of length at most $\dim_{\mathbb{C}} S$. Because $sl(2, \mathbb{C})$ is semi-simple and U/J is finite dimensional, U/J splits as a sum of simple modules each of which is isomorphic to S by hypothesis. Thus $U/J \cong S^{(n)}$ for some $n \in \mathbb{N}$, and consequently $P \cdot (U/J) = 0$. That is, $P \subseteq J$, and the conclusion follows.

LEMMA 3.2. *Let M be a finitely generated U-module of Krull dimension* 1. *Then $GK(M) \geqslant 2$.*

Proof. By [6, Chap. 2] M has a 1-critical factor module (i.e., a factor module of Krull dimension 1, any proper factor of which is artinian), and now by Corollary 2.2(i) it is enough to prove the result when M is 1-critical. Suppose M is 1-critical. If there exists an infinite chain $M = M_0 \supseteq M_1 \supseteq \cdots$ of non-zero submodules such that each M_i/M_{i+1} is infinite dimensional (thus of GK-dimension at least 1) then $GK(M) \geqslant 2$ by Corollary 2.2(ii) and we are finished. Suppose this is not the case. Then there exists a non-zero submodule M' of M such that every proper factor module of M' is finite dimensional. Furthermore M' may be chosen cyclic, and it is enough to prove that $GK(M') \geqslant 2$ in order for the lemma to hold. This is what will be proved. Let I be a left ideal such that U/I is 1-critical and every proper factor of U/I is finite dimensional.

We show first that U/I has simple factor modules of arbitrarily large finite dimension. Suppose, to the contrary, that there is $n \in \mathbb{N}$ such that every simple module of U/I has dimension $\leqslant n$. There are of course (up to isomorphism) only finitely many simple modules of dimension $\leqslant n$. Pick a chain $U = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I$ of left ideals such that each factor I_j/I_{j+1} is simple. It follows that for some sufficiently large j, the composition series for U/I_{j+1} contains at least $(n+1)$ distinct copies of the same simple module, S, say. Now the fact that U/I_{j+1} is semi-simple (being finite dimensional) implies the existence of some left ideal $J \supseteq I_{j+1}$ such that U/J is of length at least $(n+1)$ and each simple module appearing in the composition series for U/J is isomorphic to S. This contradicts Lemma 3.1 as $\dim_{\mathbb{C}} S \leqslant n$. Thus, the claim holds.

We will now show that either $I \cap \mathbb{C}[e, Q] = 0$ or $I \cap \mathbb{C}[f, Q] = 0$, whence the result will follow by Lemma 2.4. Suppose to the contrary that there are non-zero elements $p_1 = p_1(e, Q) \in I \cap \mathbb{C}[e, Q]$ and $p_2 = p_2(f, Q) \in I \cap \mathbb{C}[f, Q]$. Let n_1 denote the degree of p_1 as a polynomial in e, and let n_2 denote the degree of p_2 as a polynomial in f. It is easy to see that there exists an integer m, such that if $n \in N$ and $n \geqslant m$ then $p_1(e, n(n+2))$ and $p_2(f, n(n+2))$ have degree n_1 and n_2, respectively. Let K be a maximal left ideal of U such that $I \subseteq K$ and $\dim_{\mathbb{C}}(U/K) > m$. Put $n+1 = \dim_{\mathbb{C}}(U/K)$. Now, by the comments at the beginning of Section 3, $Q - n(n+2)$ annihilates the simple module U/K, and so $Q - n(n+2) \in K$. Because p_1, $p_2 \in I$ it follows that both $q_1 = p_1(e, n(n+2))$ and $q_2 = p_2(f, n(n+2))$ are elements of K, and non-zero. Looking at U/K as $\mathbb{C} \oplus \mathbb{C}f \oplus \cdots \oplus \mathbb{C}f^n$, there is a non-zero element $a \in U/K$, $a = \alpha_s f^s + \cdots + \alpha_t f^t$ with $s \leqslant t$, $\alpha_s \neq 0$, $\alpha_t \neq 0$ and $K = \operatorname{ann}(a)$. Consequently, $q_1 \cdot a = q_2 \cdot a = 0$. By considering the lowest degree term in $q_2 \cdot a$ it is clear that $s + \deg q_2 \geqslant n+1$. By considering the highest degree term in $q_1 \cdot a$ it is clear that $t \leqslant \deg q_1$. Hence

$$\deg q_1 + \deg q_2 \geqslant t - s + (n+1) \geqslant n+1.$$

But $\deg q_1 + \deg q_2 = n_1 + n_2$, and so $n_1 + n_2 \geqslant n+1$. But n_1 and n_2 are fixed while n can be arbitrarily large — this contradiction completes the proof of the lemma.

THEOREM 3.3. *The Krull dimension of* $U(sl(2))$ *is two.*

Proof. The result of Nouazé–Gabriel shows that $|U| \geqslant 2$, and the reverse inequality is obtained from Lemma 2.3 and Lemma 3.2 (because $GK(U) = 3$).

REFERENCES

1. D. ARNAL ET G. PINCZON, Idéaux à gauche dans les quotients simples de l'algèbre enveloppante de $sl(2)$, *Bull. Soc. Math. France.* **101** (1973), 381–395.
2. I. N. BERNSTEIN, The analytic continuation of generalized functions with respect to a parameter, *Funkcional. Anal. i Prilozen.* **6** (1972), 26–40.
3. W. BOHRO AND H. KRAFT, Über die Gelfand-Kirillov-Dimension, *Math. Ann.* **220** (1976), 1–24.
4. J. DIXMIER, Quotients simples de l'algèbre enveloppante de $sl(2)$, *J. Algebra* **24** (1973), 551–564.
5. J. DIXMIER, "Enveloping Algebras," North–Holland, Amsterdam, 1977.
6. R. GORDON AND J. C. ROBSON, Krull dimension, *Mem. Amer. Math. Soc.* **133** (1973).

7. A. JOSEPH, A generalization of Quillen's lemma and its application to the Weyl Algebras, *Israel J. Math.* **28** (1977), 177–192.
8. A. JOSEPH, Towards the Jantzen conjecture, *Comp. Math.* **40** (1980), 35–67.
9. Y. NOUAZÉ ET P. GABRIEL, Idéaux Premiers dans l'algèbre enveloppante d'une algèbre de lie nilpotente, *J. Algebra* **6** (1967), 77–99.
10. J.-E. ROOS, Compléments à l'étude des quotients primitifs des algèbres enveloppantes des algbres de lie semi-simples, *C.R. Acad. Sci. Oaris Ser. A* **276** (1973), 447–450.
11. S. P. SMITH, The primitive factor rings of the enveloping algebra of $sl(2, \mathbb{C})$, *J. London Math. Soc.*, to appear.